Improved Performance of a Digital Phase-Locked Loop Combined with a Frequency/Frequency-Rate Estimator

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When a digital phase-locked loop with a long loop update time tracks a signal with high doppler, the demodulation losses due to frequency mismatch can become very significant. One way of reducing these doppler-related losses is to compensate for the doppler effect using some kind of frequency-rate estimator. The performance of the fixed-window least-squares estimator and the Kalman filter is investigated; several doppler compensating techniques are proposed. It is shown that the variance of the frequency estimator can be made as small as desired, and with this, the doppler effect can be effectively compensated. The remaining demodulation losses due to phase jitter in the loop can be less than 0.1 dB.

I. Introduction

Figure 1 depicts the major components of a digital phase-locked loop (DPLL) used in tracking low signal-to-noise ratio (SNR) signals. The integrate-and-dump circuit integrates the phase error at the output of the phase detector producing the so-called error signal. At the loop update instants, this error signal is sampled, filtered, and used to set the frequency of the digitally controlled oscillator (DCO) to its new value. So, the continuously changing frequency of the tracked signal is followed by the DCO in a staircase fashion.

In selecting the optimum loop update time, T, one encounters two conflicting effects. On the one hand, the phase jitter in the loop decreases in proportion to $1/T$. On the other hand, the static phase error due to frequency mismatch increases in proportion to T. During high doppler rates, the demodulation losses due to the static phase error may be very significant. Figure 10[1] illustrates our point.

At the Deep Space Network (DSN) stations, the static phase error of an analog PLL is reduced by ramping the voltage-controlled oscillator (VCO) using a predicted trajectory file and the Programmed Oscillator Control Assembly (POCA). A similar technique of ramping the DCO's frequency to reduce the phase error due to doppler rate in a DPLL is considered in this analysis. However, no predicted trajectory files to do this DCO ramping will be assumed in this article. Instead, techniques of estimating the frequency of the tracked signal using linear estimators is investigated.

[1]From Simon, M., and A. Mileant, "DSA's Subcarrier Demodulation Losses," IOM 3395-85-55. Jet Propulsion Laboratory, Pasadena, Calif., March 1985. (JPL Internal document.)

In Sections II and III, the equations and performance of two linear estimators are derived, namely, of the "fixed-window" least-squares estimator and the Kalman filter. In Section V, several possible implementations of the combined estimator/DPLL demodulator are compared. It is shown that the incorporation of an estimator in the carrier/subcarrier demodulation process can virtually eliminate the doppler-related demodulation losses. Finally, in Appendix A, the transfer functions for the estimators are given for future reference.

II. Fixed-Window Least-Squares Estimator

Because of the Doppler effect, the instantaneous frequency $x(t)$ tracked by a DPLL is a time-varying function, which in the Taylor series expansion is of the form

$$x(t) = x_1 + x_2 (t - T_r) + x_3 (t - T_r)^2/2 + \cdots \tag{1}$$

where T_r is some arbitrary reference time. In this analysis, it will be assumed that only the first *two* parameters, i.e., the frequency at time T_r, x_1, and the frequency rate x_2, have significant value and need to be estimated from the available data.

At the loop update instants t_k, the loop produces $\hat{x}(k)$, which is the estimate of $x(k)$, at the DCO's output.[2] Since the DCO does not have any offsets or frequency drifts, there is a one-to-one relation between the DCO's input, $y(k)$, and its output, $x(k)$; i.e., $x(k) = cy(k)$, where c is a constant denoting the DCO gain. Without loss of generality, from now on we will assume that $c = 1$, so that $x(k) = y(k)$. Using M frequency samples, we want to estimate x_1, the frequency at time T_r, and x_2, the frequency rate in the time interval $T_r \leqslant t \leqslant MT + T_r$.

At time instants $t_k = kT$ we obtain the DCO's frequency sample, $y(k)$, which is assumed to be of the following mathematical form (see Fig. 2):

$$y(k) = x_1 + x_2 (t_k - T_r) + \nu(k) \tag{2}$$

Here $\nu(k)$ is the noise due to the phase jitter in the DPLL. It is modeled as a Gaussian random variable with zero mean and variance σ_y^2. T is the loop update time and $k = 1, 2, \cdots M$.

Given M noisy frequency samples, $y(k)$, x_1 and x_2 can be estimated using the following least-squares algorithm (Ref. 1):

$$\mathbf{x} = (\mathbf{H}^T\mathbf{H})^{-1}\, \mathbf{H}^T \mathbf{y} \tag{3}$$

[2]For convenience, we shall denote sampled values of a process $x(t)$ by $x(k)$ rather than $x(t_k)$.

where $\mathbf{x}$ is a two-dimensional estimation vector defined as

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \tag{4}$$

$\mathbf{y}$ is an M-dimensional data vector

$$\mathbf{y} = \begin{bmatrix} y(1) \\ y(2) \\ \cdot \\ \cdot \\ y(M-1) \\ y(M) \end{bmatrix} \tag{5}$$

and $\mathbf{H}$ is an $M\times 2$ observation matrix, which by inspection of Eq. (2), is

$$\mathbf{H}^T = \begin{bmatrix} 1 & 1 & \ldots\ldots\ldots\ldots & 1 \\ (T - T_r) & (2T - T_r) & \ldots\ldots & (MT - T_r) \end{bmatrix} \tag{6}$$

Using Eq. (6), it can be shown that

$$(\mathbf{H}^T\mathbf{H})^{-1} = \frac{12}{T^2 M(M-1)(M+1)} \times \begin{bmatrix} \left[T_r^2 - T_r T(M+1) + \dfrac{T^2 (M+1)(2M+1)}{6}\right] & T_r - \dfrac{T(M+1)}{2} \\ T_r - \dfrac{T(M+1)}{2} & 1 \end{bmatrix} \tag{7}$$

The above equation simplifies considerably when the reference time T_r is set to zero (equivalent to reducing the Taylor series of Eq. (1) to a Maclaurin series). In this case, the above equation becomes

$$(\mathbf{H}^T\mathbf{H})^{-1} = \frac{2}{M(M-1)} \begin{bmatrix} (2M+1) & \dfrac{-3}{T} \\ \dfrac{-3}{T} & \dfrac{6}{T^2(M+1)} \end{bmatrix} \tag{8}$$

With $T_r = 0$ and inserting Eqs. (5), (6), and (8) into Eq. (3), $\hat{x}_1$ and $\hat{x}_2$ are computed as follows

$$\mathbf{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \frac{2}{M(M-1)}$$

$$\times \begin{bmatrix} (2M+1) & \frac{-3}{T} \\ \frac{-3}{T} & \frac{6}{T^2(M+1)} \end{bmatrix} \begin{bmatrix} \sum_{k=1}^{M} y(k) \\ T \sum_{k=1}^{M} k y(k) \end{bmatrix} \qquad (9)$$

It can be shown that both $\hat{x}_1$ and $\hat{x}_2$ are *unbiased* estimates of x_1 and x_2, respectively.

The covariance matrix of the error in the estimator, according to Ref. 1, because of the independence of the noise samples $\nu(k)$, is

$$E[\mathbf{e}\mathbf{e}^T] \triangleq \mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \sigma_y^2 (\mathbf{H}^T\mathbf{H})^{-1} \qquad (10)$$

where the error vector $\mathbf{e}$ is defined as

$$\mathbf{e} \triangleq \begin{bmatrix} \hat{x}_1 - x_1 \\ \hat{x}_2 - x_2 \end{bmatrix} \qquad (11)$$

$(\mathbf{H}^T\mathbf{H})^{-1}$ is given by Eq. (7) or Eq. (8) and σ_y^2 is the variance of the DCO's frequency samples.

Observing Eq. (8), we see that for large M, the error variance for x_1 is proportional to $1/M$ whereas for x_2 the error variance is proportional to $1/M^3$. This implies that the uncertainty in the estimate of x_2 diminishes much more rapidly than for x_1.

Using the estimates $\hat{x}_1$ and $\hat{x}_2$, the frequency estimate at time t will be

$$\hat{x}(t) = \hat{x}_1 + (t - T_r)\hat{x}_2 \qquad (12)$$

with expected value

$$E[x(t)] = x_1 + (t - T_r)x_2 \qquad (13)$$

and variance

$$\mathrm{var}\,[\hat{x}(t)] \triangleq \sigma_e^2(t)$$

$$= p_{11} + 2(t - T_r)p_{12} + (t - T_r)^2 p_{22}$$

$$= \sigma_y^2 \frac{2}{M(M-1)} \left[(2M+1) - \frac{6}{T}(t - T_r) + \frac{6(t - T_r)^2}{T^2(M+1)} \right] \qquad (14)$$

Note that the variance of the estimated frequency, $\sigma_e^2(t)$, is a parabola with minimum value at time

$$t_{min} = T(M+1)/2 + T_r \qquad (15)$$

which lies in the *midpoint* of the data stream. The above equations (12) through (14) are true for prediction $(t > MT + T_r)$ as well as for smoothing $(t < MT + T_r)$.

Let γ be defined as the ratio of the variance of the frequency estimator to the variance of the frequency samples, i.e.,

$$\gamma \triangleq \frac{\sigma_e^2(t)}{\sigma_y^2} \qquad (16)$$

and again without loss of generality let $T_r = 0$. Then it can be shown that

$$\frac{1}{M} \leqslant \gamma \leqslant \frac{4M+2}{M(M-1)} \triangleq \gamma_{max} \qquad (17)$$

for $T(M+1)/2 \leqslant t \leqslant T(M+1)$.

The above equation says that our estimator will have minimum variance σ_y^2/M when smoothing is performed in the middle of the data stream. On the other hand, if we want to use the estimator as a *predictor* T seconds ahead of the most recent data point, then the variance of the estimator will be $\gamma_{max}\,\sigma_y^2$, where γ_{max} will be equal to the upper bound of Eq. (17). To improve demodulation, we want the variance of the predictor to be less then the variance of the samples, i.e., we want $\gamma_{max} < 1$. This sets the lower bound on the number of samples needed for computation of $x(t)$, namely, $M \geqslant 6$. Of course, the larger M is, the lower will be the error variance with the penalty of bigger computational burden.

In the proposed implementation of the least-squares algorithm, each time a new frequency sample becomes available, the oldest sample is discarded. This can be accomplished with a shift register as shown in Fig. 3. In this implementation, in each loop update period the tracked frequency and frequency rate are estimated from the M last frequency samples. Hence the name "fixed-window" estimator.

III. Kalman Filter Estimator

In this section, the performance of a second-order Kalman filter for estimating the frequency and the frequency rate is investigated. The Kalman filter belongs to the class of recursive linear estimators. We begin by making several definitions. Let $\mathbf{x}(k)$ be the two-dimensional state vector defined in Eq. (4). Then, in accordance with our previous discussion, the state-space equations describing the evolution of our system from time t_k to time t_{k+1} will be (see Fig. 4)

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + w(k) \tag{18}$$

$$y(k) = \mathbf{H}\mathbf{x}(k) + \nu(k) \tag{19}$$

where $\mathbf{F}$ is a 2×2 state transition matrix given by

$$\mathbf{F} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \tag{20}$$

and $\mathbf{H}$ is a 1×2 observation matrix given by

$$\mathbf{H} = [1 \;\; 0] \tag{21}$$

and is different from $\mathbf{H}$ defined previously. $y(k)$ is again the frequency noisy sample. $w(k)$ is modeled as a stationary white noise process with covariance matrix

$$\mathbf{Q} = E\,[\mathbf{w}(k)\mathbf{w}(k)^T] = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \tag{22}$$

which takes into account the unmodeled disturbances in the system e.g., the higher order terms in Eq. (1). $\nu(k)$ is the same as before. Note that $\mathbf{F}$, $\mathbf{H}$, and $\mathbf{Q}$ are assumed to be time-invariant.

The operation of the Kalman filter is given in terms of the error covariance matrix $\mathbf{P}$ and the gain matrix $\mathbf{K}$. Referring to the time diagram of Fig. 5, the equations describing the operation of the filter at the instant when the measurement $y(k)$ becomes available are (Ref. 2):

State Estimate Update

$$\hat{\mathbf{x}}_f(k) = \mathbf{x}_p(k) + \mathbf{K}(k)\,[y(k) - \mathbf{H}\mathbf{x}_p(k)] \tag{23}$$

Error Covariance Update

$$\mathbf{P}_f(k) = [\mathbf{I} - \mathbf{K}(k)\mathbf{H}]\,\mathbf{P}_p(k) \tag{24}$$

Kalman Gain Update

$$\mathbf{K}(k) = \mathbf{P}_p(k)\mathbf{H}^T\,[\mathbf{H}\mathbf{P}_p(k)\mathbf{H}^T - \sigma_y^2]^{-1}$$

$$= \begin{bmatrix} k_1(k) \\ k_2(k) \end{bmatrix} \tag{25}$$

where the subscript p stands for predicted or extrapolated values and the subscript f denotes filtered or updated values. The corresponding extrapolation or prediction equations are

State Estimate Extrapolation

$$\mathbf{x}_p(k) = \mathbf{F}\mathbf{x}_f(k-1) \tag{26}$$

Error Covariance Extrapolation

$$\mathbf{P}_p(k) = \mathbf{F}\mathbf{P}_f(k-1)\,\mathbf{F}^T + \mathbf{Q} \tag{27}$$

To assess the performance of the Kalman filter, we need to know the steady-state value of the error covariance matrix $\mathbf{P}$ in terms of $\mathbf{Q}$ and σ_y^2. It turns out that the closed form solution of the steady-state matrix $\mathbf{P}$ (and $\mathbf{K}$) is quite difficult to derive given an arbitrary $\mathbf{Q}$ matrix, even for a second-order system. We will use the results of Ref. 3 where the steady-state covariance and gain matrices are derived for $\mathbf{Q}$ of the following form

$$\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} = \sigma_q^2\, T^2 \begin{bmatrix} T^2/3 & T/2 \\ T/2 & 1 \end{bmatrix} \tag{28}$$

In our analysis we will treat σ_q^2 as a constant which we can set to some desired value. Rewriting Eqs. (12) through (18) of Ref. 3, the elements of the steady-state filtered covariance matrix are obtained, namely,

$$\left.\begin{aligned} p_{11} &= \sigma_y^2 \left[2\alpha + 2\sqrt{\alpha}\sqrt{\alpha + r^2} - 8\sqrt{\alpha + r^2} - 8/3\right]/r^2 \\ p_{12} &= p_{21} = \sigma_y^2\, 4\left[\sqrt{\alpha + r^2} - \sqrt{\alpha}\right]/(Tr^2) \\ p_{22} &= 8\,\sigma_y^2 \left[\sqrt{\alpha} - 1\right]/(Tr)^2 \end{aligned}\right\} \qquad (29)$$

where

$$r^2 \triangleq 16\sigma_y^2/(\sigma_q^2 T^4) \qquad (30)$$

and

$$\alpha = 4/3 + 2\sqrt{r^2 + 1/3} \qquad (31)$$

When the Kalman filter is used to estimate $\hat{x}(t)$ in the time interval $t_k \leqslant t \leqslant t_k + T$ (for 'real-time' demodulation), the variance of this estimator will be (analogous to Eq. (14))

$$\sigma_e^2(t) = p_{11} + 2tp_{12} + t^2 p_{22}, \quad 0 \leqslant t \leqslant T \qquad (32)$$

Inserting Eq. (29) into Eq. (32) with $t = T$ and simplifying, we obtain the maximum value of the variance of our estimated frequency, namely,

$$\sigma_e^2\Big|_{max} \triangleq \sigma_e^2 = \sigma_y^2\left[2\alpha + 2\sqrt{\alpha^2 + \alpha r^2} - 16/3\right]/r^2 \qquad (33)$$

Defining again γ as the ratio of the input to output variances of the estimator, i.e., $\gamma = \sigma_e^2/\sigma_y^2$, the curve of γ versus r^2 is obtained and shown in Fig. 6. From that figure we see that $\gamma < 1$ for $r^2 > 300$. Since r^2 is a scaled version of the ratio of the measurement variance, σ_y^2 to σ_q^2, the upper bound for choosing the latter should be (from Eq. (30))

$$\sigma_q^2 < 16\sigma_y^2/(300T^4) \qquad (34)$$

The lower bound for σ_q^2 is determined by the unaccounted dynamics of the tracked signal such as the frequency acceleration. In estimating $\mathbf{x}(k)$, the matrix $\mathbf{Q}$ has the effect of "washing out" the old data: the larger $\mathbf{Q}$ is, the less effect the old measurements will have against new ones. Conversely, small $\mathbf{Q}$ makes the estimate insensitive to new measurements. This is undesirable when significant signal dynamics are expected. The $\mathbf{Q}$ matrix also determines the size of the error in the estimator (see Eq. (27): the larger $\mathbf{Q}$ is, the larger $\mathbf{P}$ will be. The optimum set of values for $\mathbf{Q}$ has to be determined possibly by simulation.

Figure 6 also shows curves of the elements of the steady-state gain matrix, k_1 and k_2 vs r^2. These gains are related to p_{11} and p_{12} as follows:

$$\begin{aligned} k_1 &= p_{11}/\sigma_y^2 \\ k_2 &= p_{12}/\sigma_y^2 \end{aligned} \qquad (35)$$

where p_{11} and p_{12} are given by Eq. (29).

IV. Some Comparisons Between the Two Estimators

Both estimators, the least-squares and the Kalman filter, belong to the class of minimum-variance, unbiased, linear estimators. The first one represents a "batch-type" approach; the second, a recursive.

On each cycle (loop update period), the least-squares algorithm requires approximately $2M$ summations and $M + 3$ multiplications, while the Kalman filter requires approximately 19 summations and 16 multiplications.

The comparison of the error variance for the two estimators is not that obvious. In the least-squares algorithm, the variance depends on M – the number of samples considered. In the Kalman filter, the variance depends on our specification of the $\mathbf{Q}$ matrix, which can be arbitrarily selected depending on how fast we want the filter to follow the new data.

The Kalman filter seems to be a more elegant approach to the estimation problem. However, the least-squares algorithm does not have the instability and divergence problems of the $\mathbf{P}$ matrix of the Kalman filter.

V. Signal Demodulation Using Frequency Rate Compensation

There are many ways in which frequency estimators can be combined with a DPLL or can become part of a DPLL in order to improve the demodulation process of a doppler distorted signal. We will compare here three implementations which we shall call the "Parallel" the "Serial" and the "Single Loop" Estimator-Added Demodulators.

The concept of a *Parallel Estimator-Added Demodulator* is represented in Fig. 7. It consists of M DPLLs and $(M - 1)$ delays. Each subsequent loop tracks the signal delayed by T seconds relative to the previous loop. At the loop update instants, the noisy frequency samples of the M DCOs are fed into the computing device that estimates $\hat{x}_1$ and $\hat{x}_2$ and from them $\hat{x}(t)$. Finally, $\hat{x}(t)$ drives the DCO, which performs the actual demodulation. In this implementation, demodulation is accomplished with an 'open-loop receiver'. Fine tuning of the phase can be done with an epoch-tracking loop, which is shown with dotted lines. This "parallel" demodulation scheme works only with the least-squares algorithm discussed in Section II. This implementation requires the minimum number of components (DPLLs and delays) when the demodulated signal comes from the middle delay where the variance of the estimator has its minimum value. Four DPLLs with three delays should be sufficient to give an $\hat{x}(t)$ with small variance.

The same demodulation results can be obtained in a more economical way with the *Serial Estimator-Added Demodulator*, which is shown in Fig. 8. Here the noisy frequency samples, $y(k)$, are obtained from a single DPLL. These samples can be fed either into the least-squares estimator or the Kalman filter where the estimated frequency $x(t)$ is computed. The final demodulation process is identical to the one described previously, i.e., it can be performed with an open-loop receiver or with the aid of an epoch-tracking loop.

Finally, in Fig. 9, the *DPLL with a Frequency Rate Compensator* concept is depicted. At loop update instants, the $y(k)$ samples are fed into either the least-squares estimator or the Kalman filter where the frequency rate $x_2(k)$ is estimated. Then $\hat{x}_2(k)$ is used to ramp the DCO between the loop update instants. This compensates for the doppler effect and reduces the loop phase error due to frequency mismatch. Since, in this implementation, the frequency rate estimator becomes part of the DPLL, the loop filter has to compensate for the poles of the estimator. Appendix A gives the transfer functions for the least-squares estimator and the Kalman filter and Figs. 11 and 12 show in a block diagram the interaction of the DPLL components with a second order estimator. However, this is done here only for future reference. A detailed analysis of a DPLL enhanced with a frequency-rate estimator will be treated in a subsequent article.

Comparing the above three schemes, we observe that the "parallel" implementation is the least economical from the standpoint of the number of components. From the standpoint of reducing the phase error in the loop, they all appear to be equal.

VI. Conclusion

When a DPLL with a long loop update time tracks a signal with high doppler, the losses due to frequency mismatch can become very significant. One way of reducing these doppler-related losses is to compensate for the frequency rate using some kind of estimator. It was shown that the variance of the estimator can be made as small as desired. In other words, the doppler effect can be effectively compensated. The remaining demodulation losses due to phase jitter in the loop will be less than 0.1 dB, as is illustrated in Fig. 10.

In Sections II and III, the performances of the fixed-window least-squares estimator and the Kalman filter are investigated. They both belong to the class of minimum-variance linear estimators. The least-squares is a batch-type algorithm, whereas the Kalman filter uses a recursive algorithm. Appendix A gives the transfer functions of these estimators for future reference. In Section V, several possible doppler compensating techniques are proposed.

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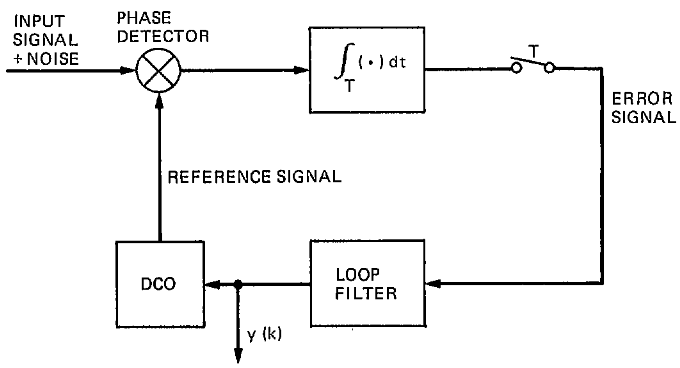

Fig. 1. Digital phase-locked loop block diagram

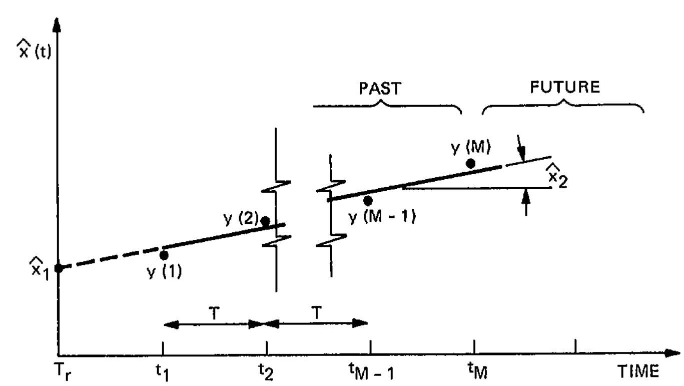

Fig. 2. Model of the frequency process

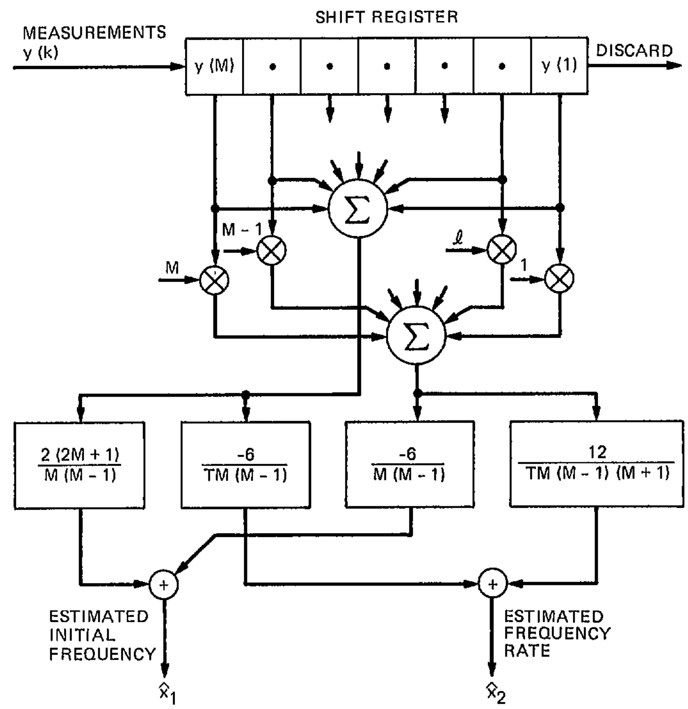

Fig. 3. Fixed-window least-squares estimator block diagram

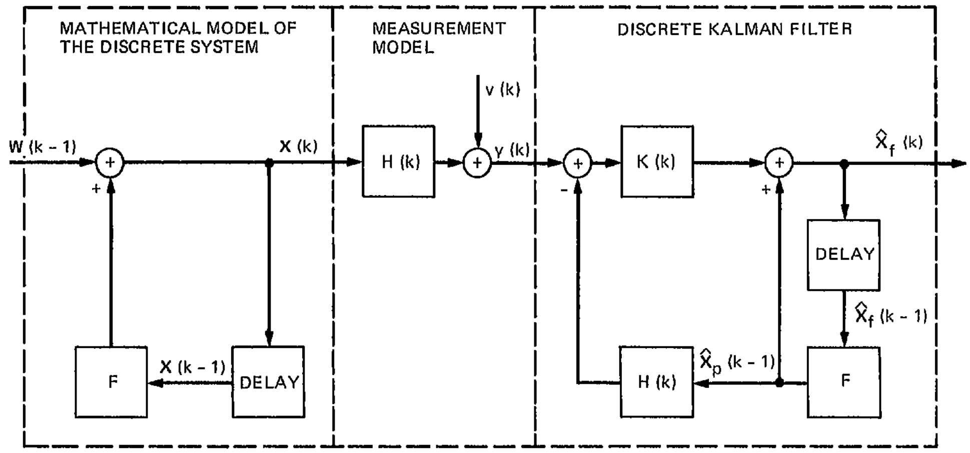

Fig. 4. Kalman filter timing diagram

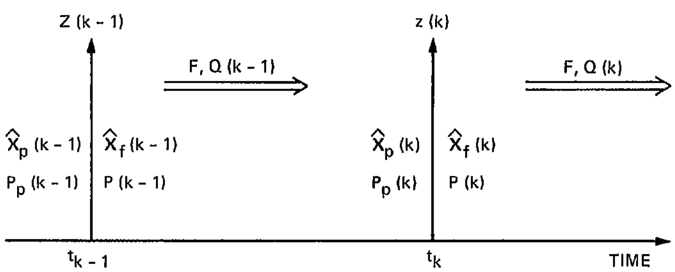

Fig. 5. System model of the discrete Kalman filter

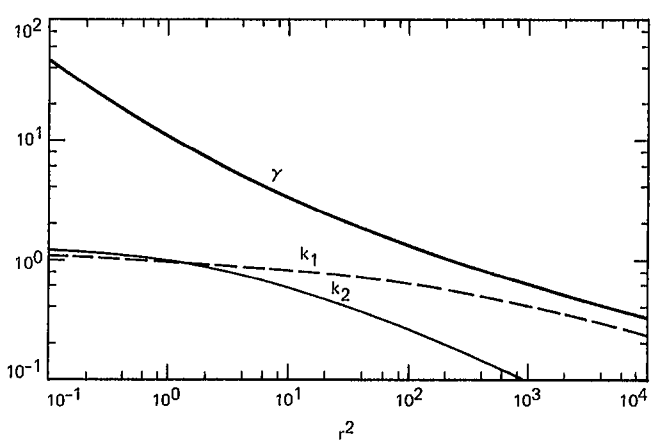

Fig. 6. Steady-state normalized variance of the Kalman filter estimated frequency and gains vs r^2

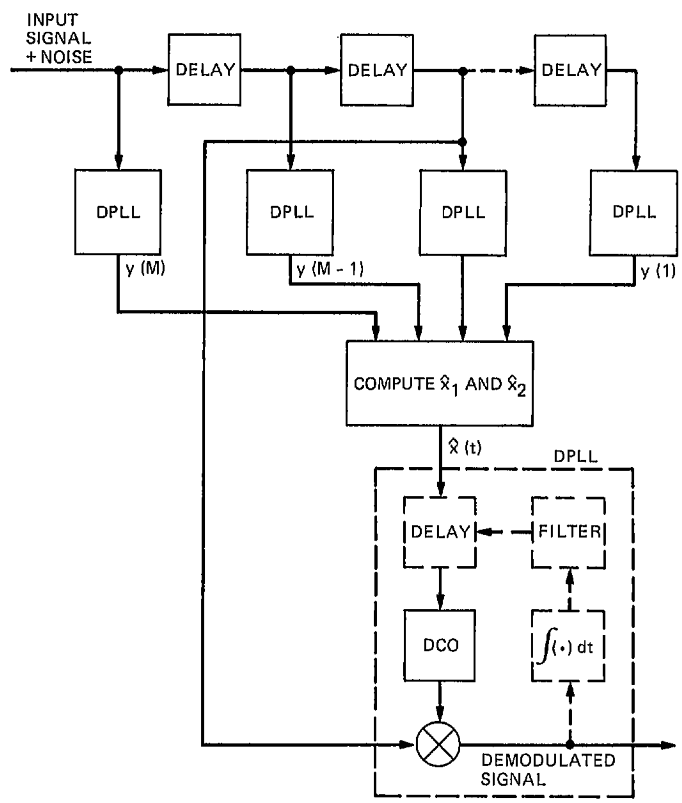

Fig. 7. Demodulation with several DPLLs — "parallel implementation"

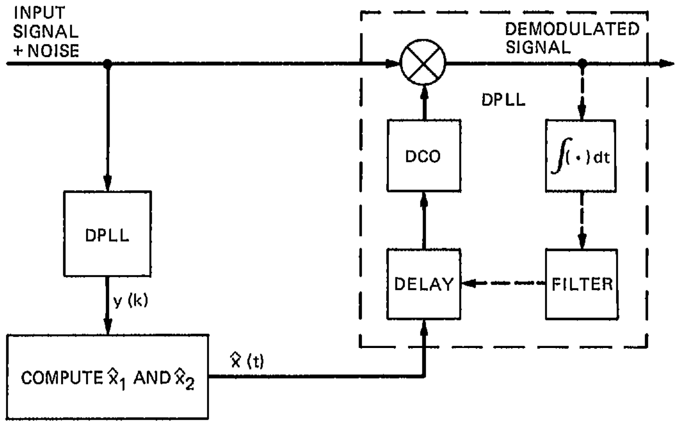

Fig. 8. Demodulation with two DPLLs—"serial implementation"

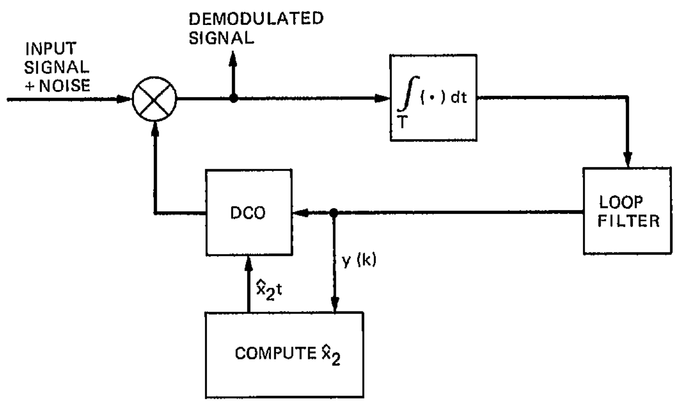

Fig. 9. DPLL with frequency rate compensation

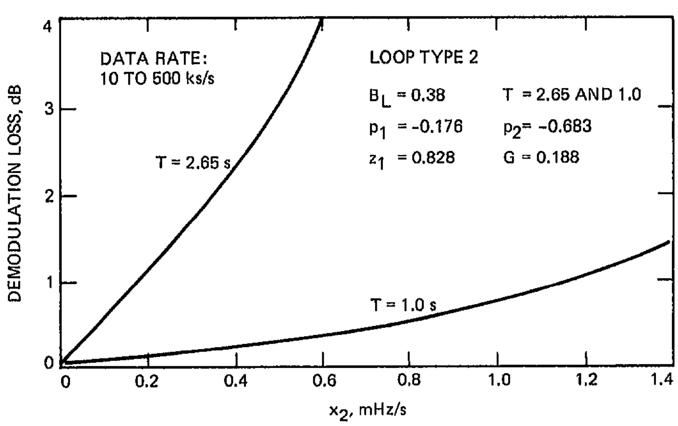

Fig. 10. Subcarrier demodulation losses vs. frequency rate (x_2)

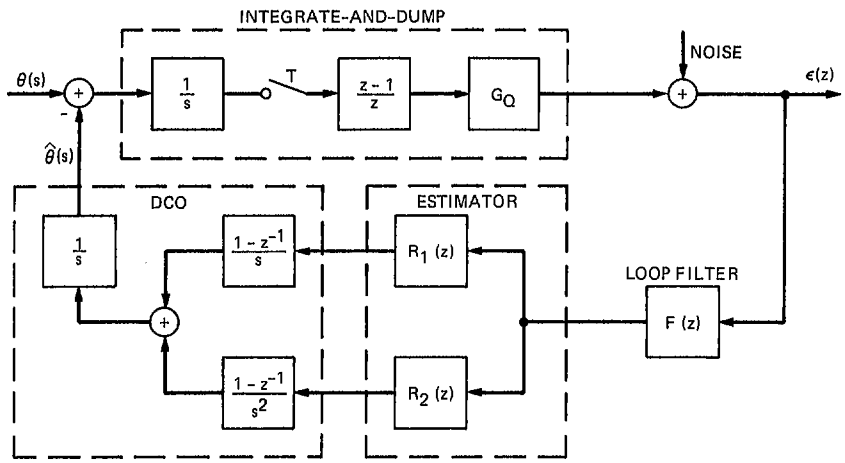

Fig. 11. Hybrid s/z diagram of a DPLL with a second-order estimator

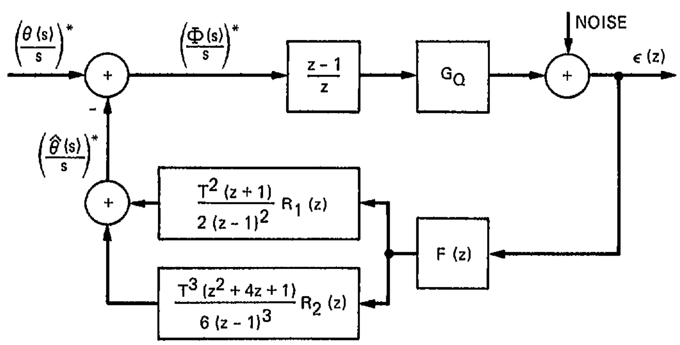

Fig. 12. Equivalent z-domain loop diagram

Appendix A

Transfer Functions for the Fixed-Window Least Squares Estimator and the Kalman Filter

The *fixed-window least squares* algorithm requires the sums

$$S_1 = \sum_{i=1}^{M} y(k) \quad \text{and} \quad S_2 = T\sum_{i=1}^{M} ky(k) \tag{A1}$$

Defining z^{-1} as the delay operator, i.e., $y(k)z^{-1} = y(k-1)$, we can write the above sums as follows

$$S_1 = y(M)\sum_{i=0}^{M-1} z^{-i} \quad \text{and} \quad S_2 = Ty(M)\sum_{i=0}^{M-1} (M-i)z^{-i} \tag{A2}$$

Performing the above summations, we get

$$S_1 = y(M)\,\frac{z^M - 1}{z^{M-1}(z-1)}$$

and

$$S_2 = Ty(M)\,\frac{Mz^{M+1} - z^M(M+1) + 1}{z^{M-1}(z-1)^2} \tag{A3}$$

Taking the z-transform of the above expressions we obtain

$$S_1(z) = Y(z)\,\frac{z^M - 1}{z^{M-1}(z-1)}$$

and

$$S_2(z) = TY(z)\,\frac{Mz^{M+1} - z^M(M+1) + 1}{z^{M-1}(z-1)^2} \tag{A4}$$

Combining Eq. (9) with the above equations, we finally obtain the desired transfer function for the fixed-window least squares estimator, namely,

$$R_1(z) \triangleq \frac{\hat{X}_1(z)}{Y(z)} = \frac{2}{M(M-1)} \times \left[\frac{(2M+1)(z^M-1)}{z^{M-1}(z-1)} - \frac{3(Mz^{M+1} - z^M(M+1)+1)}{z^{M-1}(z-1)^2}\right]$$

and

$$R_2(z) \triangleq \frac{\hat{X}_2(z)}{Y(z)} = \frac{2}{M(M-1)} \times \left[-\frac{3(z^M-1)}{Tz^{M-1}(z-1)} + \frac{6(Mz^{M+1} - z^M(M+1)+1)}{T(M+1)z^{M-1}(z-1)^2}\right] \tag{A5}$$

To obtain the transfer function of the *Kalman filter*, we insert Eqs. (26), (20), and (21) into Eq. (23) and obtain

$$\begin{bmatrix}\hat{x}_1(k)\\ \hat{x}_2(k)\end{bmatrix} = \begin{bmatrix}1 & T\\ 0 & 1\end{bmatrix}\begin{bmatrix}\hat{x}_1(k-1)\\ \hat{x}_2(k-1)\end{bmatrix} + \left\{y(k) - [1\quad 0]\begin{bmatrix}1 & T\\ 0 & 1\end{bmatrix}\begin{bmatrix}\hat{x}_1(k-1)\\ \hat{x}_2(k-1)\end{bmatrix}\right\}\begin{bmatrix}k_1(k)\\ k_2(k)\end{bmatrix} \tag{A6}$$

In steady-state, $k_1(k) = k_1$ and $k_2(k) = k_2$. Taking now the z-transform of the above equation and recombining terms, we obtain

$$\begin{aligned}\hat{X}_1(z)(1 - z^{-1}(1-k_1)) &= \hat{X}_2(z)Tz^{-1}(1-k_1) + k_1Y(z)\\ \hat{X}_2(z)(1 - z^{-1}(1-k_2T)) &= -\hat{X}_1(z)k_2z^{-1} + k_2Y(z)\end{aligned} \tag{A7}$$

Writing the above equation in matrix form and performing matrix inversion, we finally obtain the transfer function for the Kalman filter, namely,

$$R_1(z) = \frac{\hat{X}_1(z)}{Y(z)}$$

$$= \frac{z(zk_1 + k_2 T - k_1)}{z^2 + z(-2 + k_1 + k_2 T) + (1 - k_1)}$$

$$R_2(z) = \frac{\hat{X}_2(z)}{Y(z)}$$

$$= \frac{zk_2(z - 1)}{z^2 + z(-2 + k_1 + k_2 T) + (1 - k_1)} \qquad \text{(A8)}$$

The above transfer functions are needed when the least-squares estimator or the Kalman filter becomes part of a digital phase-locked loop (DPLL). For future reference, Fig. 11 shows the building blocks of a DPLL with the frequency and frequency-rate estimator incorporated in the loop. This block diagram is in the hybrid s/z transform domain. Figure 12 shows the corresponding z-domain block diagram, which is obtained from Fig. 11 using techniques of Ref. 4. In Figs. 11 and 12, zero computation time was assumed. Using techniques of Ref. 4, the loop filter $F(z)$ can be designed so that the estimator-enhanced DPLL will have optimum stability and bandwidth characteristics. All of these will be the subject of a future analysis.